FINITE MARKOV CHAINS
APPLIED TO
LIFT TRAFFIC CALCULATION AND SIMULATION

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ABSTRACT

The process why a lift car moves from a floor to another is an "stochastic process" , which mathematical treatment was begun by A.A.Markov (1856-1922), characterized by an initial state and probabilities of transition from an state or floor to another one. The whole transition probabilities among the different floors or "states" can be arranged as a "system transition matrix". The same concept can be extended to a lift group with several cars. The interest of this model lies in it can be developed by means of a powerful mathematical tool, the matricial calculus, easily computerized. On the other hand, Monte Carlo method application allows to carry out system behaviour simulations under different hypotheses. In this presentation, the author shows the essential concepts in relation to this matter as well as the wide possibilities this mathematical model offers for lift traffic calculus and simulation.

1. STOCHASTIC PROCESSES

Let us consider a system which can adopt the states E_i (i= 1, 2, 3, ... n) and so that the state changes can only be produced in specific discrete instants or stages 1, 2, 3, ... which are the stages n the system passes thorugh.

Being $p_i(n)$ the probability that the system is in the state E_i in the stage n. The probabilities $p_i(n)$ corresponding to a given stage can be represented by a vector of so many dimensions as the number of states the system can adopt. This vector is limited in wideness and direction by the condition that its components are not negative and have an addition equal to 1.

(1) $$p(n) = [\, p_1(n), p_2(n), \ldots, p_i(n), \ldots, p_n(n)]$$

A vector with the above mentioned characteristics is named "stochastic vector" and when its components represent a system possible states, it will be called system "state vector".

Let us suppose now that the transition of an state to another don't depend on more than those two states. More precisely, let us suppose that to all pair of values (E_i, E_j) can be associated a conditional probability p_{jk} in such a way that the state j corresponds to the stage n and the k one to n+1, as well as the initial probabilities $p_i(0)$. In that case we have a Markov chain, defined by the equations :

(2) $$p_k(n+1) = \Sigma\ p_j(n)\, p_{jk} \qquad (j = 0,1,2, \dots)$$

being p_{jk} the transition probabilities between the states j and the state k.

The whole of transition probabilities among states in a step, that´s to say, between two succesive system stages is the "transition matrix" of the system in one step [M] :

$$[M] = \begin{bmatrix} p_{11} & p_{12} & p_{13} & \cdots & p_{1k} & \cdots & p_{1m} \\ p_{21} & p_{22} & p_{23} & \cdots & p_{2k} & \cdots & p_{2m} \\ p_{31} & p_{32} & p_{33} & \cdots & p_{3k} & \cdots & p_{3m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{j1} & p_{j2} & p_{j3} & \cdots & p_{jk} & \cdots & p_{jm} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{m1} & p_{m2} & p_{m3} & \cdots & p_{mk} & \cdots & p_{mm} \end{bmatrix}$$

The elements of a transition matrix row are the transition probabilities from the state represented by the fixed index of the same to each one of the rest. One matrix column elements are the transition probabilities from each one of the possible states to the state represented by the fixed index of that column.

The equation (2) can be written as follows :

(3) $$[\, p_1(n+1), p_2(n+1), p_3(n+1) \dots, p_m(n+1) \,] = [\, p_1(n), p_2(n), p_3(n) \dots, p_m(n) \,]\, [M]$$

The system state vector is, therefore, equal to the product of the state vector in the previous stage per the transition matrix, that is :

(4) $$p(n+1) = p(n)\ [M]$$

The transition matrix is composed by elements p_k such as :

(5) $$0 \le p_{jk} \le 1$$

(6) $$\Sigma\ p_{jk} = 1 \quad \text{for every } j$$

The equation (5) expresses the condition that, being probabilities, the elements must have a value comprised between 0 to 1. The equation (6) expresses the condition that the addition of all row elements must be 1, representing the total probability of all possible transitions from a given stage, in one step.

2. DYNAMIC MATRIX AND EQUATION

It is named the system "dynamic matrix" to the [A] matrix, such as :

$$[A] = [M] - [1] \tag{7}$$

where [1] is the unity matrix with the same order than [M].

$$[A] = \begin{bmatrix} p_{11}-1 & p_{12} & p_{13} & \cdots & p_{1m} \\ p_{21} & p_{22}-1 & p_{23} & \cdots & p_{2m} \\ p_{31} & p_{32} & p_{33}-1 & \cdots & p_{3m} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{m1} & p_{m2} & p_{m3} & \cdots & p_{mm}-1 \end{bmatrix} \tag{8}$$

This matrix comply with the conditions :

(9) $0 \leq a_{ij} \leq 1$ $\quad -1 \leq a_{ij} \leq 0$ $\quad \Sigma\, a_{ij} = 0 \quad (j=1, 2,3, \ldots, m)$

The equation (7) can be written now in this way :

$$\mathbf{p(n)}\ [A] = \mathbf{p(n)}\ [M] - \mathbf{p(n)}\ [1]$$

$$\mathbf{p(n+1)} - \mathbf{p(n)} = \mathbf{p(n)}\ [A] \tag{10}$$

named "dynamic equation" of a Markov chain

3. HIGH ORDER TRANSITIONS

The transition probability of E_i state to E_j in a step is p_{ij} and the transition matrix from the vector **p(n)** to the vector **p(n+1)** is [M]. Let us calculate now the transition probability from E_i to E_j in two steps and the transition matrix from **p(n)** to **p(n+2)**.

The probability to pass from E_i state to E_j in two steps will be (Fig. 1) :

$$p_{ij}^{(2)} = \Sigma\, p_{ik}\, p_{kj} \qquad (k = 1, 2, \ldots, m) \tag{11}$$

Nevertheless, this expression is the product of ith row of [M] by the jth column of[M]. Therefore, the transition matrix in two steps is $[M]^2$.

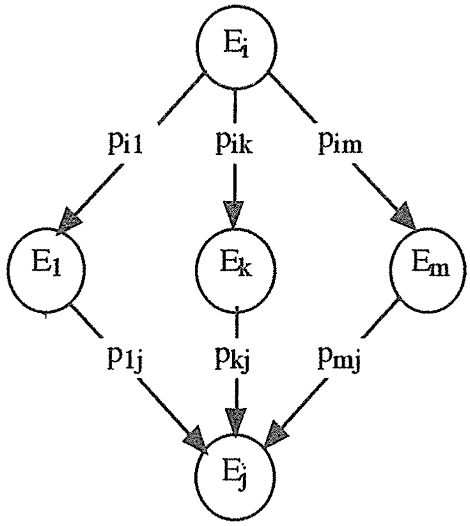

Figure 1

In a general way, the transition matrix in n steps will be :

(12) $$[M]^{(n)} = [M]^n$$

Therefore, if the system state vector in the starting time is p(0), the state vector after n steps will be :

(13) $$p(n) = p(0)\ [M]^n$$

5. ERGODIC PROPERTY

It is said that the stochastic process represented by the [M] transition matrix has the ergodic property if it is able to reach an stable state in probability, independent from the initial state.

The conditions in order to reach that limit stationary state are the transition matrix is not irreducible nor periodical, that is, must be "regular". In fact, these conditions are equivalent to the condition that one state can be always reached from whatever other state in a finite number of steps. The ergodic property means that :

(10) $$\lim_{n \to \infty} [p(n)] = [p(\infty)] = [p^*]$$

In these conditions, the state vector tends to a limit vector and the probabilities of the state tend also to their respective limit values :

(14) $$p^* = [\, p1^*, p2^*, p3^* \ldots, pm^*]$$

Besides, obviously it will be fullfilled that :

(15) $$p1^* + p2^* + p3^* + \ldots + pm^* = 1$$

To find **p*** when exists is solved the equations system given by :

(16) $$\mathbf{p}^* = \mathbf{p}^* \, [M]$$

The compatibility condition of the last equations system and, therefore, the condition that the system reaches an stationary state is that the determinant composed by the unknown coefficients is not null. However, this determinant is precisely I A I and, consequently, the stability condition is :

(17) $$| A | \neq 0$$

which justifies the [A] matrix designation as "dynamic matrix", by analogy with vibrations mechanic.

In fact, as every equation of (16) is lineal combination of the resting ones, the system composed by m-1 of these equations and the equation (15) will be solved.

This equation system resolution can be achieved by the CRAMER method and with the help of a computer.

Let us call I Bi I to the determinant formed when replacing in the determinant I A I the elements of ith row by ones.Easily can be that verified that :

(18) $$p_i^* = \frac{|B_i|}{|M|}$$

Another important feature of the ergodic chains is :

(19) $$\lim_{n \to \infty} [M]^n = [P^*]$$

where [M*] is the square matrix which lines are all the same and which elements are the components of vector **p***.

After an enough number of stages, the transition matrix between the initial state and the observed stage adopts the form of [P*], independently from the starting state.

6. AVERAGE LENGTH OF STATE CHANGES

It´s called "average length" the change from an E_i to another E_j the average number or mathematic hope of the number of necessary stages to pass from one to the other.

Being n_j the mathematic hope of the necessary step number in order to reach for the first time E_j leaving from E_i. It´s passed from E_i to E_j in one step, with the probability p_{ik}, if k=j. Let us call [N] the matrix made of the elements n_{ij} and $[N^0]$ this same matrix where the main diagonal elements have been replaced by 0.

Let us call [C] the square matrix of the same order than [N] with all elements equal to 1. According to the above mentioned , we can write :

(20) $$[N] = [M]\ [N^0] + [C]$$

This equation system can be also solved by the CRAMER method and with the help of a computer, allows to calculate n_{ij} values.

Now, let us multiply the to members of (14) by [M], hence :

(21) $$[M]^n\,[N] = [M]^{n+1}\,[N^0] + [C]$$

(22) $$[M]^n\,[C] = [M]\,[C] = [C]$$

But, if [M] is ergodic, for n -> ∞

(23) $$[M^*]\,[N] = [M^*]\,[N^0] + [C]$$

(24) $$[M^*]\,(\,[N] - [N^0]\,) = [C]$$

However, ([N] - $[N^0]$ is a diagonal matrix composed by the main diagonal elements of [N] and [M*] and [C] are matrices which lines are all identical, hence :

(25) $$[1, 1, 1, \ldots] = \left[p_1^*, p_2^*, p_3^*, \ldots\right] \begin{bmatrix} n_{11} & 0 & 0 & \ldots \\ 0 & n_{22} & 0 & \ldots \\ 0 & 0 & n_{33} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{bmatrix}$$

Therefore :

(26) $$n_{11} = \frac{1}{p_1^*} \quad n_{22} = \frac{1}{p_2^*} \quad n_{33} = \frac{1}{p_3^*} \quad \ldots$$

This means that, once the stationary state has been reached, the average stage numer to return for the first time to a given state is equal to the stationary probability inverse of that concrete state.

7. APLICATION TO LIFT TRAFFIC CALCULATION

The travels of a lift in a building can also considered as an stochastic process, within several conditioms. In this case, in each one of the leaps that it carries out between two floors, the lift system evolves from an state represented by the starting floor to another represented by the arriving floor.

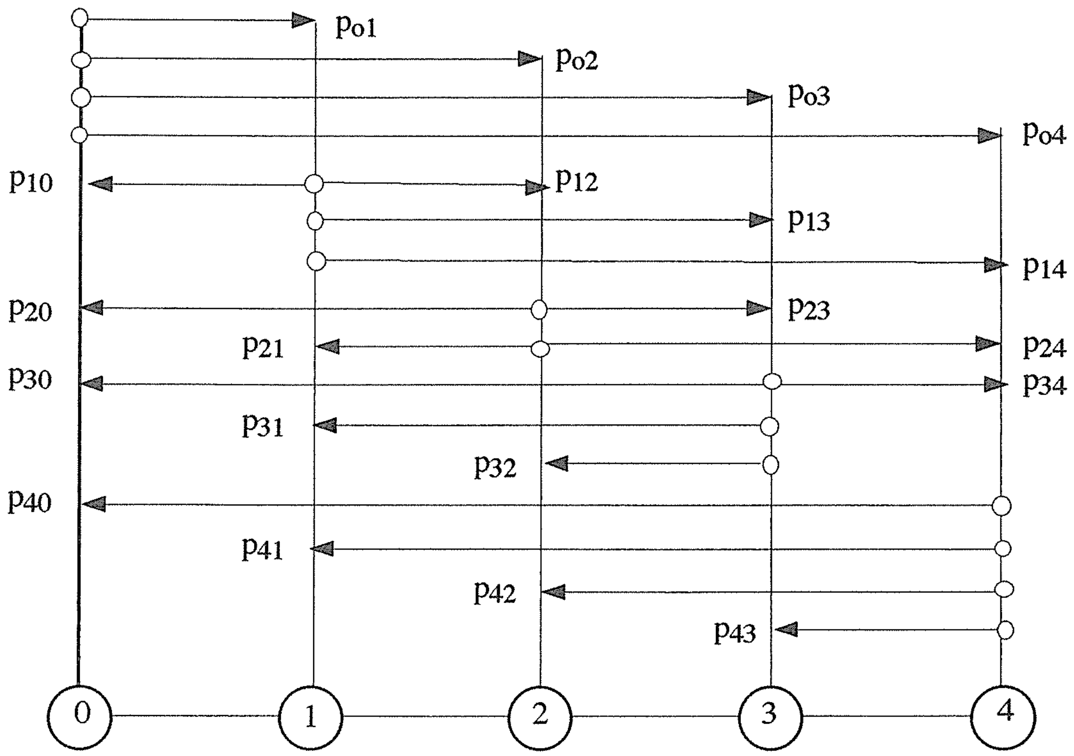

Figure 2

In figure 2 the possible one step transitions between two floors are represented for a four floor, over the main floor, building and the corresponding transition probabilities are mentioned. As transitions from one floor to itself are not possible, all the transition matrix diagonal elements will be 0 and the matrix will have N(N-1) elements or one step possible transitions, 20 in the case of this figure.

A lift vertical traffic respond to some patterns depending on the kind of demand such as intense up-peak traffic, intense down-peak traffic, inter- floors traffic, etc., which correspond to different situations that take place in the building over and therefore, can be generally identified and analysed separately. On its turn, the type of control determines the kind of possible transitions.

In general, the demand generated by passengers in each floor with concrete destination in each of the remaining floors is difficult to calculate or observate, unless very important restrictions are introduced in the calculation. However, the lift leaps between two floor can be easily registered and analysed in order to identify the corresponding traffic patterns and to elucidate the transition probabilities of the leap between every two fllors.

Once the transition matrix corresponding o a concrete traffic pattern is determined. it can be analysed from a Markov chains approach, explained previously. Since a transition probabilities statistical estimation is available, the model built in this way can be tested by Monte Carlo method, generating a lift movement simulation which can be analysed and compared with the registered one. The different existing statistical approximation techniques will allow this model improving.

Since the object of this paper is just to expound the essential concepts of the possible Markov chains aplication to lift traffic calculation, we have developed a simple instance that allows us to present this concepts.

To this aim, we consider a case of a lift intense up-peak traffic in a four floor, over the main floor, building such as the one represented in figure two, with uniform population whose transition probabilities can be easily calculated. An umiform distribution of the arrivals to the main floor, a six people car capacity, and a non external calls replying lift are assumed.

Under these conditions the transition matrix will be :

$$(27) \qquad [M] = \begin{bmatrix} 0 & p_{01} & p_{02} & p_{03} & p_{04} \\ p_{10} & 0 & p_{12} & p_{13} & p_{14} \\ p_{20} & 0 & 0 & p_{23} & p_{24} \\ p_{30} & 0 & 0 & 0 & p_{34} \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The results referred to the transition probabilities p_{ij} , the average number of stops and the highest reversal floor are shown in Table 1.

Table 1
UP-PEAK TRAFFIC
Floors : 0 + 4 Car capacity : 6 persons

		n	Πpij	n Πpij
[1]	**Transition probabilities**			
	P01		0,82202	
	P02		0,16235	
	P03		0,01538	
	P04		0,00024	
	P10		0,00024	
	P12		0,80249	
	P13		0,18212	
	P14		0,01514	
	P20		0,05387	
	P23		0,79916	
	P24		0,14697	
	P30		0,16240	
	P34		0,83760	
	P40		1,00000	
[2]	**Node's equations**			
	P01+P02+P03+P04		1,0000	
	P10+P12+P13+P14		1,0000	
	P20+P23+P24		1,0000	
	P30+P34		1,0000	
[3]	**Paths probabilities**	n	Πpij	n Πpij
1	0->1->0	1	0,0002	0,0002
2	0->2->0	1	0,0087	0,0087
3	0->3->0	1	0,0025	0,0025
4	0->4->0	1	0,0002	0,0002
5	0->1->2->0	2	0,0355	0,0711
6	0->1->3->0	2	0,0243	0,0486
7	0->1->4->0	2	0,0124	0,0249
8	0->2->3->0	2	0,0211	0,0421
9	0->2->4->0	2	0,0239	0,0477
10	0->3->4->0	2	0,0129	0,0258
11	0->1->2->3->0	3	0,0856	0,2568
12	0->1->2->4->0	3	0,0970	0,2909
13	0->1->3->4->0	3	0,1254	0,3762
14	0->2->3->4->0	3	0,1087	0,3260
15	0->1->2->3->4->0	4	0,4416	1,7663
	Mean number of stops (S)			3,2881
[4]	**Highest reversal floor**			
	1	1	0,0002	0,0002
	2	2	0,0443	0,0886
	3	3	0,1335	0,4005
	4	4	0,8220	3,2881
	H			3,7773

Transition probabilities are partly calculated in a direct way and partly by means of the equations that express that the probabilities addition of all possible leaps from a floor is 1 as well as the equations that express that probabilities of reaching each of the floors from the main floor, thorugh every possible pathway are identical since the building population is uniform. A calculation by Monte Carlo method would be better. To this end, we would generate random arrays of six numbers with the numbers 1,2, 3, 4. After we will identify the corresponding lift path for each array and the probabilities of each possible path and each possible one step transition, hence all possible p_{ij}.

The average number of stops (S) is the mean of the probabilities of each pathway leaving from the main floor weighed to the lenght of those pathways. The highest reversal floor is the average of the probabilities of every possible return pathway to the main floor weighed to the lenght of those pathways. In this way, S = 3,29 and H=3,78 are obtained. By means of traditional traffic calculatton expressions, S =3,29 and H=3,80 would have been obtained.

Solving the system formed by equations (15) and (16) the stationary state vector is obtained and in this case it is :

(30) $p^* = [\ p_0^*,\ p_1^*,\ p_2^*,\ p_3^*,\ p_4^*\ ] = [\ 0{,}24,\ 0{,}19,\ 0{,}19,\ 0{,}19,\ 0{,}19]$

This means that in an stationary order , the lift will travel to floor 0 in 24% of its stops and to each of the floors ranging from 1 to 4 in 19% of its stops. .

The inverses of the different states stationary probabilities will provide us with the average number of leaps between two visits to the same floor.

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BIOGRAPHICAL NOTE

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